Applications of Solvable Structures to the Nonlocal Symmetry-Reduction of Odes

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APPLICATIONS OF SOLVABLE STRUCTURES
TO THE NONLOCAL SYMMETRY-REDUCTION OF ODEs

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An application of solvable structures to the reduction of ODEs with a lack of local symmetries is given. Solvable structures considered here are all defined in a nonlocal extension, or covering space, of a given ODE. Examples of the reduction procedure are provided.

Keywords: Solvable structures; nonlocal symmetries; covering; ordinary differential equations; reduction.

1. Introduction

In the last few decades a renewed interest was devoted to pioneering works of Lie and Cartan, in particular for what concern the application of group theory to the differential equations and the symmetry reduction theory.

It is well known that the knowledge of a solvable k-dimensional algebra of symmetries, for a k-order ordinary differential equation (ODE) $\mathcal{E}$, guarantees that $\mathcal{E}$ can be completely integrated by quadratures [14, 15, 29]. On the other hand, finding all local symmetries for a given ODE is not always possible and one may encounter equations which are integrable by quadratures but with a lack of local symmetries. Examples of this kind are well known in recent literature, see for example [8–10, 17, 18, 22]. In fact local symmetries (classical or higher) of a k-order ODE $\mathcal{E}$ in the unknown u are described by the solutions of a linear partial differential equation (PDE) depending on the derivative of u up to the order $k-1$. Since the general solution of this PDE cannot be found unless one already knows the general solution of $\mathcal{E}$, one usually only searches for particular solutions depending on derivative of u up to order $k-2$. Therefore, in practice, $\mathcal{E}$ could not be completely reduced by quadratures

if it does not admit a solvable k-dimensional algebra of such computable symmetries. For these reasons local symmetries are sometimes inadequate and various attempts for a more effective symmetry-reduction method have been proposed. Among these, in this paper we consider the notion of λ-*symmetry* introduced in [22] and that of *solvable structure* introduced in [4, 27]. We propose here a reduction method which takes advantage of both notions.

The relevance of λ-symmetries is due to the fact that many equations, which do not possess Lie point symmetries, admit λ-symmetries and can be reduced as in the case of standard symmetries [13, 16, 17, 21, 23, 26]. Despite their name, however, λ-symmetries are neither Lie point nor higher symmetries.

As shown in [10], λ-symmetries of an ODE $\mathcal{E}$ can be interpreted as shadows of some nonlocal symmetries. In practice it means that, by embedding $\mathcal{E}$ in a suitable system $\mathcal{E}'$ determined by the function λ, any λ-symmetry of $\mathcal{E}$ can be recovered as a local symmetry of $\mathcal{E}'$.

This interpretation of λ-symmetries has many advantages and will be one of the main ingredients of this paper.

For what concerns the notion of solvable structure, this was introduced around 1990 by Basarab-Horwath and further investigated by Sherring and Prince in the papers [4–6, 27]. Solvable structures are defined by systems of vector fields, not necessarily symmetries, which provide a noteworthy generalization of the standard symmetry reduction method for completely integrable distributions.

Despite their relevance, however, solvable structures have not received so much attention and only few papers have been published on the subject (see [11, 19]). In part, this was certainly due to computational problems with the determining equations of solvable structures. In fact, these equations are in general very complicated and fully nonlinear. Nevertheless, in some cases and making use of modern symbolic manipulation packages (see [11]), one can overcome these difficulties and hence succeed in the application of the method.

A noteworthy simplification, in practice, may come by computing solvable structures which are adapted to admitted symmetry algebras, if any. Hence one certainly takes advantage of the presence of any kind of symmetry. In particular, one can include the nonlocal symmetries corresponding to λ-symmetries. This kind of (nonlocal) solvable structures were already considered in the paper [11]; in this paper we mainly review the results of [11] and further provide more examples of the resulting reduction scheme.

In our opinion, solvable structures deserve much more attention and we hope that this paper, together with [11], will strengthen the current research interests in this topic. The paper is organized as follows. In Sec. 2, we collect all notations and basic facts we need on symmetries (local and nonlocal) of differential equations. In particular we recall the interpretation of λ-symmetries as shadows of nonlocal symmetries. In Sec. 3, we recall the basic facts we need on solvable structures in a form which is suitable to our further discussion. Then, in Sec. 4, we discuss our reduction scheme of ODEs via nonlocal solvable structures. In particular, we collect here some examples which illustrate the application of the method.

2. Preliminaries

In this section we collect some notations and basic facts we need in the paper. The reader is referred to [7, 8, 15, 24, 25, 28, 29] for further details.

2.1. *ODEs as submanifolds of jet spaces*

Let M and E be smooth manifolds and $\pi : E \to M$ be a q-dimensional bundle. We denote by $\pi_k : J^k(\pi) \to M$ the k-order *jet bundle* associated to π and by $j_k(s)$ the k-order *jet prolongation* of a section s of π. Since in this paper we are only concerned with the case dim $M = 1$, we assume that M and E have local coordinates x and $(x, u^1, \dots, u^q)$, respectively. Correspondingly, the induced natural coordinates on $J^k(\pi)$ will be (x, u_i^a), $1 \le a \le q$, $i = 0, 1, \dots, k$, where u_i^a denotes the ith derivative of u^a. Moreover, when no confusion arises, Einstein summation convention over repeated indices will be used.

The *k-order jet space* $J^k(\pi)$ is a manifold equipped with the smooth distribution $\mathcal{C}^k$ of tangent planes to graphs of k-order jet prolongations $j_k(s)$. This is the *contact (or Cartan) distribution of $J^k(\pi)$*.

In this framework a k-th order system of differential equations can be regarded as a submanifold $\mathcal{E} \subset J^k(\pi)$ and any solution of the system is a section of π whose k-order prolongation is an integral manifold of the restriction $\mathcal{C}^k|_\mathcal{E}$ of the contact distribution to $\mathcal{E}$. In this paper we will deal only with (systems of) ordinary differential equations $\mathcal{E}$ which are in *normal form* and *not underdetermined*.

Analogously to $J^k(\pi)$, we can define the bundle of infinite jets $J^\infty(\pi)$: this is an infinite dimensional manifold with induced coordinates (x, u_i^a), $1 \le a \le q$, $i = 0, 1, \dots$. The set $D(\pi)$ of vector fields on $J^\infty(\pi)$ has the structure of a Lie algebra, with respect to the usual Lie bracket [12, 24, 29]. Moreover one can also define a *contact distribution* $\mathcal{C}$ on $J^\infty(\pi)$, generated by the *total derivative* operator

$$D_x = \partial_x + u_{i+1}^a \partial_{u_i^a}.$$

Notice that $\mathcal{C}$ is the annihilator space of the *contact forms* $\theta_s^a = du_s^a - u_{s+1}^a dx$, i.e. the space of vector fields such that $X \lrcorner \theta_s^a = 0 \; \forall \, a, s$ (here $\lrcorner$ denotes the insertion operator).

Given a kth order differential equation $\mathcal{E} = \{F = 0\}$, the *lth prolongation of $\mathcal{E}$* is the submanifold $\mathcal{E}^{(l)} := \{D_x^s(F) = 0 : s = 0, 1, \dots, l\}$. Analogously, we define the *infinite prolongation* $\mathcal{E}^{(\infty)}$.

2.2. *Local symmetries*

Given a smooth distribution $\mathcal{D}$ on $J^k(\pi)$, a vector field X defined on $J^k(\pi)$ is an infinitesimal symmetry of $\mathcal{D}$ if and only if $L_X \mathcal{D} \subseteq \mathcal{D}$, where L_X denotes the Lie derivative along X. The infinitesimal symmetries of $\mathcal{C}^k$ are called *Lie symmetries* of $J^k(\pi)$. These symmetries can be divided in two classes (see [24, 29]): *Lie point symmetries*, which are obtained by prolonging vector fields X on E, and *Lie contact symmetries*, which are obtained by prolonging vector fields X on $J^1(\pi)$.

By definition, symmetries of $\mathcal{C}^k$ which are tangent to $\mathcal{E}$ are called *classical symmetries of $\mathcal{E}$*.

An analogous geometric picture holds on the infinite jet spaces. However, contrary to the case of finite order jet spaces, symmetries of $\mathcal{C}$ cannot always be recovered by infinite prolongations of Lie symmetries. In fact it can be proved that $X = \xi \partial_x + \eta_i^a \partial_{u_i^a}$ is an infinitesimal symmetry of $\mathcal{C}$ if and only if $\xi, \eta^a \in \mathcal{F}(\pi)$ are arbitrary functions and

$$\eta_i^a = D_x(\eta_{i-1}^a) - u_i^a D_x(\xi), \quad \eta_0^a = \eta^a.$$

Hence, X is the infinite prolongation of a Lie point (or contact) symmetry iff ξ, η_0^a are functions on E (or $J^1(\pi)$, respectively).

On $J^\infty(\pi)$, symmetries X of $\mathcal{C}$ which are tangent to $\mathcal{E}^{(\infty)}$ are called *higher symmetries* of $\mathcal{E}$ and are determined by the condition $X(F)|_{\mathcal{E}^{(\infty)}} = 0$.

In particular in this paper we only consider symmetries in the so called *evolutive form*, i.e., symmetries of the form $X = D_x^i(\varphi^a)\partial_{u_i^a}$, $\varphi^a := \eta^a - u_1^a\xi$. The functions φ^a are called the *generating functions* (or characteristics) of X. Moreover, for the applications we are interested in, we only need to consider symmetries of $\mathcal{C}$ which are tangent to $\mathcal{E}^{(\infty)}$. This choice turns out to be also convenient since it noteworthy simplifies computations (see [29] and [2] fore more details and other aspects of ∞-jets theory).

2.3. *Nonlocal symmetries and λ-symmetries*

In recent years many authors have proposed various generalizations of notion of symmetry and some new classes of symmetries have been introduced. Among these generalizations there are those introduced by Muriel and Romero in [22] and known as λ-symmetries.

In our opinion (see [10]) a good framework to deal with many of these generalizations is the nonlocal setting. We follow here the approach to nonlocality based on the theory of *coverings* (see [20] and also [29]). However, since we only deal with ODEs and in this paper we do not want to go into geometrical details, our approach will be drastically simplified. The interested reader is referred to [29] for the general theory of coverings and nonlocal symmetries.

Roughly speaking, given a k-th order ODE

$$\mathcal{E} := \{u_k = f(x, u, u_1, \ldots, u_{k-1})\}, \tag{2.1}$$

a one-dimensional covering for $\mathcal{E}$ is the infinite prolongation $(\mathcal{E}')^\infty$ of a system $\mathcal{E}'$ of the form

$$\begin{cases} u_k = f(x, u, u_1, \ldots, u_{k-1}), \\ w_1 = H(x, u, u_1, w) \end{cases} \tag{2.2}$$

involving a new variable w.

Nonlocal symmetries of $\mathcal{E}$ are the symmetries of the system $(\mathcal{E}')^\infty$ and they have the form

$$Y = \xi\partial_x + \eta_i\partial_{u_i} + \psi_i\partial_{w_i} \tag{2.3}$$

with $\eta_i = \widetilde{D}_x(\eta_{i-1}) - \widetilde{D}_x(\xi)u_i$ and $\psi_i = \widetilde{D}_x(\psi_{i-1}) - w_i\widetilde{D}_x(\xi)$, where $\widetilde{D}_x = \partial_x + u_1\partial_u + \cdots + f\partial_{u_{k-1}} + H\partial_w$ is the restriction to $(\mathcal{E}')^\infty$ of the total derivative operator.

In this paper we limit to a particular kind of nonlocal symmetry occurring in literature, which is known as λ-symmetry.

Despite their name, λ-symmetries are neither Lie symmetries nor higher symmetries of $\mathcal{E}$. Nevertheless, as discussed in [10], λ-symmetries can be interpreted as shadows of nonlocal symmetries. More precisely, $\mathcal{E}$ admits a λ-symmetry X iff $\mathcal{E}' = \{u_k = f, w_1 = \lambda\}$, with $\lambda \in C^\infty(\mathcal{E})$, admits a (higher) symmetry with generating functions of the form $\varphi^\alpha = e^w\varphi_0^\alpha(x, u, u_1, \ldots, u_{k-1})$, $\alpha = 1, 2$.

3. Solvable Structures

In this section we will recall basic definitions and facts on (local) solvable structures in the form we need in our study. The reader is referred to [4,5] and [27] for further details.

The original purpose of solvable structures was to generalize the integrating factor approach and the classical theorem concerning reduction of order of ODEs possessing a solvable algebra of point symmetries.

Since in this paper we restrict our attention to the integration of 1-dimensional distributions, we will summarize the results of [4] and [5,27] only in this case.

Given a 1-dimensional distribution $\mathcal{D} = \langle Z \rangle$, on an n-dimensional manifold N, the definition of a solvable structure for $\mathcal{D}$ is the following

Definition 1. The vector fields $\{Y_1, \ldots, Y_{n-1}\}$ on N form a solvable structure for $\mathcal{D} = \langle Z \rangle$ if and only if, denoting $\mathcal{D}_0 = \mathcal{D}$ and $\mathcal{D}_h = \langle Z, Y_1, \ldots, Y_h \rangle$, the following two conditions are satisfied:

(i) $\mathcal{D}_{n-1} = TN$;
(ii) $L_{Y_h} \mathcal{D}_{h-1} \subset \mathcal{D}_{h-1}, \ \forall h \in \{1, \ldots, n-1\}$.

The main difference with the definition of a solvable symmetry algebra of $\mathcal{D}$ is that the fields belonging to a solvable structure do not need to be symmetries for $\mathcal{D}$. This fact represents an advantage since it gives more freedom in the choice of the fields one can use in the integration of $\mathcal{D}$.

Remark 1. It is straightforward, by the definition above, that in principle a solvable structure for $\mathcal{D}$ always exists in a neighborhood of a nonsingular point for $\mathcal{D}$. In fact, if $\{x^i\}$ is a local chart on N such that $Z = \partial_{x^1}$, one can simply consider the solvable structure generated by ∂_{x^j}, $j = 2, \ldots, n$. This structure is in particular an Abelian algebra of symmetries for $\mathcal{D}$. Nevertheless, for a given distribution $\mathcal{D}$, it is difficult to find explicitly such a local chart.

In the case of 1-dimensional distributions, the main result of [4] can be stated as follows

Proposition 1. *Let $\{Y_1, \ldots, Y_{n-1}\}$ be a solvable structure for a 1-dimensional distribution $\mathcal{D} = \langle Z \rangle$ on an orientable n-dimensional manifold N. Then, for any given volume form Ω on N, $\mathcal{D}$ can be described as the annihilator space of the system of 1-forms $\{\omega_1, \ldots, \omega_{n-1}\}$ defined as*

$$\omega_i = \frac{1}{\Delta}(Y_1 \lrcorner \cdots \lrcorner \widehat{Y_i} \lrcorner \cdots \lrcorner Y_{n-1} \lrcorner Z \lrcorner \Omega), \qquad (3.1)$$

where the hat denotes omission of the corresponding field, $\lrcorner$ denotes the insertion operator and Δ is the function defined by

$$\Delta = Y_1 \lrcorner Y_2 \lrcorner \cdots \lrcorner Y_{n-1} \lrcorner Z \lrcorner \Omega.$$

Moreover, one has that

$$d\omega_{n-1} = 0,$$
$$d\omega_i = 0 \quad mod\{\omega_{i+1}, \ldots, \omega_{n-1}\}$$

for any $i \in \{1, \ldots, n-2\}$.

The reader is referred to [4] or [5] for the proof.

It follows that, under the assumptions of this proposition, $\mathcal{D}$ can be completely integrated by quadratures (at least locally). In fact, in view of $d\omega_{n-1} = 0$, locally $\omega_{n-1} = dI_{n-1}$ for some smooth function I_{n-1}. Now, since $d\omega_{n-2} = 0 \mod\{\omega_{n-1}\}$, ω_{n-2} is closed on the level manifolds of I_{n-1}. Then, iterating this procedure, it is possible to compute all the integrals $\{I_1, \ldots, I_{n-1}\}$ of $\mathcal{D}$ and eventually find its (local) integral manifolds in implicit form $\{I_1 = c_1, \ldots, I_{n-1} = c_{n-1}\}$.

Proposition 1 can also be applied to the integration of ODEs. In fact, in view of above discussion, one can think of an ODE $\mathcal{E}$ of the form (2.1) as a manifold equipped with the 1-dimensional distribution $\mathcal{D} = \langle \overline{D}_x \rangle$, where $\overline{D}_x = \partial_x + u_1 \partial_u + \cdots + f \partial_{u_{k-1}}$ is the restriction to $\mathcal{E}$ of the total derivative operator. In particular, equation $\mathcal{E}$ can be seen as a $(k+1)$-dimensional submanifold of $J^k(\pi)$ naturally equipped with the volume form

$$\Omega = dx \wedge du \wedge \cdots \wedge du_{k-1}, \tag{3.2}$$

where $\wedge$ denotes the usual wedge product.

Then, by applying Proposition 1 to ODEs one readily gets the following:

Proposition 2. *Let $\{Y_1, \ldots, Y_k\}$ be a solvable structure for the 1-dimensional distribution $\mathcal{D} = \langle \overline{D}_x \rangle$ on $\mathcal{E}$ defined by (2.1). Then $\mathcal{E}$ is integrable by quadratures and the general solution of $\mathcal{E}$ can be obtained in implicit form by subsequently integrating the system of one forms $\omega_k, \ldots, \omega_1$, in the given order.*

4. Applications of Solvable Structures in the Integration of ODEs with a Lack of Lie Point Symmetries

The main point of Proposition 2 is that the knowledge of a solvable structure for an ODE $\mathcal{E}$ allows one to completely integrate it, even though $\mathcal{E}$ admits only a one dimensional symmetry algebra. However, we have already remarked that finding solvable structures is not always easy and in practice a noteworthy simplification may come by computing solvable structures adapted to an admitted symmetry algebra $\mathcal{G}$, if any. In fact, as discussed in [11], under this assumption the determining equation of a solvable structure extending $\mathcal{G}$ become more reasonable. It follows that one certainly takes advantage of the presence of any kind of symmetry. In particular one can include the nonlocal symmetries corresponding to λ-symmetries and one can integrate, for example, second order ODEs which admits λ-symmetries but without Lie point symmetries. In this case, if Y is a nonlocal symmetry corresponding to a λ-symmetry, one has $[\partial_w, Y] = Y$ and it suffices to search for solvable structures which extend the 2-dimensional algebra spanned by ∂_w and Y.

On the other hand, if one considers a general covering system $\mathcal{E}'$, it is not true in general that $\mathcal{E}'$ inherits the local symmetries of $\mathcal{E}$. In fact this only happens if H is a joint invariant for the local symmetries of $\mathcal{E}$. When $\partial_w H = 0$, however, any algebra $\mathcal{G}$ of symmetries for $\mathcal{E}$ is always an algebra of symmetries for the distribution $\langle \widetilde{D}_x, \partial_w \rangle$. Hence, if the algebra $\mathcal{G} = \langle Y_1, Y_2, \ldots, Y_r \rangle_{\mathbb{R}}$ (given by the $\mathbb{R}$-linear span of the fields $Y_1, \ldots, Y_r$) is solvable, one may just consider solvable structures which extend $\{\partial_w, Y_1, Y_2, \ldots, Y_r\}$.

Another useful simplification may occur when one only knows $\{Y_1, \ldots, Y_r\}$, $r \leq k - 1$, and a complete system of joint invariants $\{\gamma_1, \ldots, \gamma_{k-r}\}$ for the distribution $\mathcal{D}_r$. In fact, using the same notations of Definition 1 and Proposition 1 one has the following (see [11]).

Proposition 3. *Let $\mathcal{D} = \langle Z \rangle$ be a 1-dimensional distribution and $\{Y_1, \ldots, Y_r\}$, $r \leq k - 1$, be such that $L_{Y_h}\mathcal{D}_{h-1} \subset \mathcal{D}_{h-1}$ for any $h \in \{1, \ldots, r\}$. If one knows a complete system of joint invariants $\{\gamma_1, \ldots, \gamma_{k-r-1}\}$ for the distribution $\mathcal{D}_r$, then $\mathcal{D}$ is completely integrable by quadratures.*

4.1. Examples

If an ODE $\mathcal{E}$ admits a solvable structure, application of Proposition 1 to the integration of $\mathcal{E}$ makes use of subsequent integration of some systems of closed 1-forms Σ_i defined on some manifolds M_i. Here some preliminary remarks, on the practical application of this procedure, are in order.

In general, due to the topology of manifolds M_i, this procedure returns only local pieces for solutions of $\mathcal{E}$. Indeed global solutions, in general, must be constructed by glueing toghether local pieces of solutions defined on corresponding overlapping domains.

Moreover, in practice, one often get into solvable structures which are well defined with the only exception of a closed thin subset of $\mathcal{E}$. For example, a typical case is that of a solvable structure generated by a system of vector fields which are linearly dependent on some lower dimensional submanifold Γ of $\mathcal{E}$. In such a cases, above procedure can only be applied to compute generic solutions, that is solutions which do not intersect Γ. A different choice of the solvable structure can be used to describe the missed solutions, if any.

The following examples provide an insight into the applications of above reduction scheme. All the examples make use of solvable structures adapted to symmetries of the given ODE. In particular, since in a solvable structure there is no need to distinguish between symmetries and other vector fields, all these vector fields will be treated on the same footing.

Example 1. Consider the ODE $\mathcal{E}$ defined by

$$u_2 = -\frac{x^2}{4u^3} - u - \frac{1}{2u}, \tag{4.1}$$

where $u \neq 0$.

As shown in [22], (4.1) has no point symmetries but admits a λ-symmetry with $\lambda = x/u^2$. If we consider the system

$$\begin{cases} u_2 = -\dfrac{x^2}{4u^3} - u - \dfrac{1}{2u}, \\ w_1 = \dfrac{x}{u^2} \end{cases} \tag{4.2}$$

a nonlocal symmetry Y of (4.1) which corresponds to the λ-symmetry found by Muriel and Romero in [22] is that generated by the functions $\varphi^1 = ue^w$ and $\varphi^2 = -2e^w$. We will search for solvable structures which extend the nonabelian algebra $\mathcal{G} = \langle Y_1 = Y, Y_2 = \partial_w \rangle$.

By making use of the Maple 11 routines, it is not difficult to find the most general solvable structure $\{Y_1, Y_2, Y_3\}$ for $\mathcal{D} = \langle Y_0 \rangle$ (recall that, in this case, Y_0 is the total derivative operator restricted to (4.2)). The determining equations $L_{Y_3}(\mathcal{D}_2) \subset \mathcal{D}_2$, admit the solution

$Y_3 = a_1 \partial_x + a_2 \partial_u + a_3 \partial_{u_1} + a_4 \partial_w$, with

$$a_1 = \frac{(4u^4 + 4u^2 u_1^2 + 4xuu_1 + x^2)F + 8u((uu_1 + x)a_2 - u^2 a_3)}{8u^4 + u^2(8u_1^2 + 4) + 8xuu_1 + 2x^2},$$

a_2, a_3, a_4 arbitrary functions of (x, u, u_1, w) and F an arbitrary function of $x + \arctan((2uu_1 + x)/(2u^2))$. However, to give an illustration of above integration procedure, we just consider the following particular solution (with $F = 2$, $a_2 = a_4 = 0$ and $a_3 = -1/(2u)$)

$$Y_3 = \partial_x - \frac{1}{2u} \partial_{u_1}.$$

In this case one can check that $\Delta = Y_1 \lrcorner Y_2 \lrcorner Y_3 \lrcorner \bar{D}_x \lrcorner \Omega$ vanishes iff

$$4u^2 u_1^2 + 4xuu_1 + 4u^4 + x^2 = 0. \tag{4.3}$$

Then $\{Y_1, Y_2, Y_3\}$ is a solvable structure only at points of $\mathcal{E}$ which do not belong to the submanifold Γ defined by (4.3). We apply above integration procedure to $M_1 = \mathcal{E} \backslash \Gamma$.

Under these assumptions, since both $\{Y_1, Y_2, Y_3\}$ and $\{Y_1, Y_3, Y_2\}$ are solvable structures, one can check that $d\omega_3 = d\omega_2 = 0$. Hence, one finds $\omega_3 = dI_3$, $\omega_2 = dI_2$ with

$$I_2 = 2 \ln |u| - w - \ln |4u^2 u_1^2 + 4xuu_1 + 4u^4 + x^2|,$$

$$I_3 = \arctan \left(\frac{u^2 u_1 + xu/2}{u^3} \right) + x.$$

It follows that, when restricted on the level manifolds $M_2 = M_1 \cap \{I_2 = c_2, I_3 = c_3\}$, ω_1 is an exact 1-form and one can check that it is the exterior derivative of the function

$$I_1 = \frac{2e^{c_2} u^2}{\cos(x - c_3)^2} + 2e^{c_2} \ln |\cos(c_3 - x)| - 2e^{c_2} x \tan(c_3 - x).$$

Hence one gets that on M_1 the general solution of (4.2) can be written in the implicit form $\{I_1 = c_1, I_2 = c_2, I_3 = c_3\}$. However, since one can easily solve with respect to u, one gets

$$u = \pm \cos(x - c_3) \sqrt{\frac{c_1 e^{-c_2}}{2} - x \tan(x - c_3) - \ln |\cos(x - c_3)|}.$$

This solution depends on 3 arbitrary constants, but of course one of them is inessential. In fact, by suitably rearranging constants c_i's, one can write this solution in the form

$$u = \pm \cos(x - C_2) \sqrt{C_1 - x \tan(x - C_2) - \ln(\cos(x - C_2))}, \quad C_1, C_2 \in \mathbb{R}. \tag{4.4}$$

This way we obtain the most general solutions which lie on $\mathcal{E}_+ = M_1 \cap \{u > 0\}$ and $\mathcal{E}_- = M_1 \cap \{u < 0\}$.

Solutions (4.4) are generic in the sense that they do not intersect the submanifold Γ. Special solutions which intersect Γ, if any, should be computed by using a different solvable structure. However, by using a different integration procedure [22], one can show that there are no special solutions and indeed (4.4) is the general integral of (4.1).

Example 2. Consider the ODE $\mathcal{E}$ defined by

$$u_2 = \frac{u_1^2}{u} + e^x \frac{u_1}{u} + xu - e^x, \tag{4.5}$$

where $u \neq 0$. Equation (4.5) has no point symmetries but admits an exponential nonlocal symmetry in the λ-covering with $\lambda = e^x/u$. In fact, by considering the covering system $\mathcal{E}'$ defined by (4.5) together with $w_1 = \lambda$, one gets a symmetry Y generated by $\varphi^1 = -ue^w$ and $\varphi^2 = e^w$. Here we use a solvable structure to get the complete integration of (4.5).

Examples of solvable structures $\{Y_1, Y_2, Y_3\}$ which extend the nonabelian algebra $\mathcal{G} = \langle Y_1 = Y, Y_2 = \partial_w \rangle$ can be easily computed by solving the determining equations $L_{Y_3}(\mathcal{D}_2) \subset \mathcal{D}_2$. In this case we will just use the one defined by

$$Y_3 = \frac{u^2}{2(u_1 + e^x)} \partial_u - \frac{u}{u_1 + e^x} \partial_w.$$

Since $\{Y_1, Y_2, Y_3\}$ is a solvable structure only at points of $\mathcal{E}$ which do not belong to the submanifold Γ defined by $u_1 + e^u = 0$, our computation are limited to the submanifold $M_1 = \mathcal{E} \backslash \Gamma$. Here, one finds that $\omega_3 = dI_3$ with

$$I_3 = -\frac{2u_1}{u} - \frac{2e^x}{u} + x^2.$$

Then, on the level manifolds $M_2 = M_1 \cap \{I_3 = c_3\}$, ω_1 is an exact 1-form and one gets that $\omega_2 = dI_2$ with

$$I_2 = -w - \ln|u| + \frac{x^3}{6} - \frac{c_3 x}{2}.$$

Finally, on $M_3 = M_2 \cap \{I_3 = c_3, I_2 = c_2\}$, one has $\omega_1 = dI_1$ with

$$I_1 = \exp\left(-\frac{1}{6}x^3 + \frac{1}{2}c_3 x + c_2\right) u + \int \exp\left(x - \frac{1}{6}x^3 + \frac{1}{2}c_3 x + c_2\right) dx.$$

Hence, under above assumptions, the generic solution of (4.5) can be written in the form

$$u = -\frac{\displaystyle\int \exp\left(x - \frac{1}{6}x^3 + \frac{1}{2}Cx\right) dx}{\exp\left(-\frac{1}{6}x^3 + \frac{1}{2}Cx\right)}, \quad C \in \mathbb{R}$$

where we have absorbed an arbitrary constant in the indefinite integral and conveniently rearranged the remaining one.

Example 3. Consider the ODE $\mathcal{E}$ defined by

$$u_2 = e^u u_1 + x. \tag{4.6}$$

Even though (4.6) can be easily reducible, since it can be written in the form

$$D_x\left(u_1 - e^u - \frac{x^2}{2}\right) = 0,$$

one can easily check that (4.6) has no point symmetries. Equation (4.6), however, admits an exponential nonlocal symmetry in the λ-covering with $\lambda = e^u$. In fact, by considering the

covering system $\mathcal{E}'$ defined by (4.6) together with $w_1 = \lambda$, one gets a symmetry Y generated by $\varphi^1 = e^w$ and $\varphi^2 = e^w$. Here we use a solvable structure to get the complete integration of (4.6).

Examples of solvable structures $\{Y_1, Y_2, Y_3\}$ which extend the nonabelian algebra $\mathcal{G} = \langle Y_1 = Y, Y_2 = \partial_w \rangle$ can be easily computed by solving the determining equations $L_{Y_3}(\mathcal{D}_2) \subset \mathcal{D}_2$. In this case we will just use the one globally defined by

$$Y_3 = \partial_{u_1}.$$

Using this structure, one finds that $\omega_3 = dI_3$ with

$$I_3 = u_1 - \frac{x^2}{2} - e^u.$$

Then, on the submanifolds $\{I_3 = c_3\}$, one gets that $\omega_2 = dI_2$ with

$$I_2 = u - w - \frac{x^3}{6} - c_3 x.$$

Finally, on the submanifolds $\{I_3 = c_3, I_2 = c_2\}$, $\omega_1 = dI_1$ with

$$I_3 = \exp\left(-u + \frac{x^3}{6} + c_3 x + c_2\right) + \int \exp\left(\frac{x^3}{6} + c_3 x + c_2\right) dx.$$

Hence, the general solution of (4.6) can be written in the form

$$u = \frac{x^3}{6} + Cx - \ln\left(-\int \exp\left(\frac{x^3}{6} + Cx\right) dx\right), \quad C \in \mathbb{R}$$

where we have absorbed an arbitrary constant in the indefinite integral and conveniently rearranged the remaining one.

Example 4. Consider the ODE $\mathcal{E}$ defined by

$$u_2 = (e^u + 1)u_1 - e^u + x. \tag{4.7}$$

Equation (4.7) has no point symmetries but admits an exponential nonlocal symmetry in the λ-covering with $\lambda = e^u$. In fact, by considering the covering system $\mathcal{E}'$ defined by (4.7) together with $w_1 = \lambda$, one gets a symmetry Y generated by $\varphi^1 = e^w$ and $\varphi^2 = e^w$. Here we use a solvable structure to get the complete integration of (4.7).

Examples of solvable structures $\{Y_1, Y_2, Y_3\}$ which extend the nonabelian algebra $\mathcal{G} = \langle Y_1 = Y, Y_2 = \partial_w \rangle$ can be easily computed by solving the determining equations $L_{Y_3}(\mathcal{D}_2) \subset \mathcal{D}_2$. In this case we will just use the one globally defined by

$$Y_3 = e^{x-u}\partial_u.$$

Using this structure, one finds that $\omega_3 = dI_3$ with

$$I_3 = e^{-x}(e^u - 1 - x - u_1).$$

Then, on the submanifolds $\{I_3 = c_3\}$, one gets that $\omega_2 = dI_2$ with

$$I_2 = u - w + x + \frac{x^2}{2} + e^x c_3.$$

Finally, on the submanifolds $\{I_3 = c_3, I_2 = c_2\}$, $\omega_1 = dI_1$ with

$$I_3 = \exp\left(-u - x - \frac{x^2}{2} - c_3 e^x + c_2\right) + \int \exp\left(-x - \frac{x^2}{2} - c_3 e^x + c_2\right) dx.$$

Hence, the general solution of (4.7) can be written in the form

$$u = -x - \frac{x^2}{2} - Ce^x - \ln\left(-\int \exp\left(-x - \frac{x^2}{2} - Ce^x\right) dx\right), \quad C \in \mathbb{R}$$

where we have absorbed an arbitrary constant in the indefinite integral and conveniently rearranged the remaining one.

Example 5. Consider the ODE $\mathcal{E}$ defined by

$$u_2 = \frac{u_1^2}{u} + u_1\left(\frac{x}{u^3} + \frac{1}{x}\right) + xu, \tag{4.8}$$

where $xu \neq 0$. Equation (4.8) has no point symmetries but admits an exponential nonlocal symmetry in the λ-covering with $\lambda = x/u^3$. In fact, by considering the covering system $\mathcal{E}'$ defined by (4.8) together with $w_1 = \lambda$, one gets a symmetry Y generated by $\varphi^1 = -ue^w/3$ and $\varphi^2 = e^w$. Here we use a solvable structure to get the complete integration of (4.8).

Examples of solvable structures $\{Y_1, Y_2, Y_3\}$ which extend the nonabelian algebra $\mathcal{G} = \langle Y_1 = Y, Y_2 = \partial_w \rangle$ can be easily computed by solving the determining equations $L_{Y_3}(\mathcal{D}_2) \subset \mathcal{D}_2$. In this case we will just use the one globally defined by

$$Y_3 = xu\partial_{u_1}.$$

Using this structure, one finds that $\omega_3 = dI_3$ with

$$I_3 = \frac{u_1}{xu} - x + \frac{1}{3u^3}.$$

Then, on the submanifolds $\{I_3 = c_3\}$, one gets that $\omega_2 = dI_2$ with

$$I_2 = -w - 3\ln|u| + x^3 + \frac{3}{2}c_3 x^2.$$

Finally, on the submanifolds $\{I_3 = c_3, I_2 = c_2\}$, $\omega_1 = dI_1$ with

$$I_1 = u^3 \exp\left(-x^3 - \frac{3}{2}c_3 x^2 + c_2\right) + \int \exp\left(-x^3 - \frac{3}{2}c_3 x^2 + c_2\right) xdx.$$

Hence, the general solution of (4.8) can be written in the form

$$u = -\left[\exp\left(-x^3 - \frac{3}{2}Cx^2\right)\left(\int \exp\left(-x^3 - \frac{3}{2}Cx^2\right) xdx\right)\right]^{1/3}, \quad C \in \mathbb{R}$$

where we have absorbed an arbitrary constant in the indefinite integral and conveniently rearranged the remaining one.

Example 6. Consider the ODE $\mathcal{E}$ defined by

$$u_2 = (xu_1 - xu^2 + u^2)e^{-1/u} + \frac{2u_1^2}{u} + u_1, \tag{4.9}$$

where $u \neq 0$. It can be shown that (4.9) has no point symmetries but admits an exponential nonlocal symmetry in the λ-covering with $\lambda = xe^{-1/u} - 1/x$. In fact, by considering the covering system $\mathcal{E}'$ defined by (4.9) together with $w_1 = \lambda$, one gets a nonlocal symmetry Y generated by $\varphi^1 = e^w$ and $\varphi^2 = e^w$. Here we use a solvable structure to get the complete integration of (4.9).

Examples of solvable structures $\{Y_1, Y_2, Y_3\}$ which extend the nonabelian algebra $\mathcal{G} = \langle Y_1 = Y, Y_2 = \partial_w \rangle$ can be easily computed by solving the determining equations $L_{Y_3}(\mathcal{D}_2) \subset \mathcal{D}_2$. In this case we will just use the one globally defined by

$$Y_3 = u^2 e^x \partial_{u_1}.$$

Using this structure, and proceeding like in the previous examples, one finds that the general solution of (4.9) can be written in the form

$$u = \frac{1}{Ce^x + \ln\left(\int \frac{-x}{e^{Ce^x}}dx\right)}, \quad C \in \mathbb{R}$$

where we have absorbed an arbitrary constant in the indefinite integral and conveniently rearranged the remaining one.

Example 7. Consider the ODE $\mathcal{E}$ defined by

$$u_2 = -\frac{x}{u}u_1 - \frac{x^2}{u} + xu - 1, \tag{4.10}$$

where $u \neq 0$. Equation (4.10) has no point symmetries but admits an exponential nonlocal symmetry in the λ-covering with $\lambda = x/u$. In fact, by considering the covering system $\mathcal{E}'$ defined by (4.10) together with $w_1 = \lambda$, one gets a symmetry Y generated by $\varphi^1 = -ue^w$ and $\varphi^2 = e^w$.

In this case, equations $L_{Y_3}(\mathcal{D}_2) \subset \mathcal{D}_2$ admit particular solutions in terms of Airy functions $Ai(x)$ and $Bi(x)$, which form a pair of linearly independent solutions of the ODE $w'' + xw = 0$ (see [3]). For example, a solvable structure $\{Y_1, Y_2, Y_3\}$ which extends the nonabelian algebra $\mathcal{G} = \langle Y_1 = Y, Y_2 = \partial_w \rangle$ is that defined by

$$Y_3 = \frac{(Ai(1,x)u - (u_1 + x)Ai(x))^2}{u(-Ai(x)Bi(1,x) + Bi(x)Ai(1,x))^2}\partial_{u_1} + \frac{2Ai(x)(Ai(1,x)u - (u_1 + x)Ai(x))}{u(-Ai(x)Bi(1,x) + Bi(x)Ai(1,x))^2}\partial_w.$$

where $Ai(1,x)$ and $Bi(1,x)$ denote first order derivatives $D(Ai)(x)$ and $D(Bi)(x)$, respectively. Using such a solvable structure, however, ω_3 is hardly integrable. That is why we will follow a different approach.

As already remarked in Proposition 3, in the reduction of integrable distributions one can take advantage of joint scalar invariants. This example can be used to clarify this point.

To this end we will make the following particular choice of a joint differential invariant

$$I = \frac{-Bi(1,x)u + (u_1 + x)Bi(x)}{Ai(1,x)u - (u_1 + x)Ai(x)}$$

for $\{\tilde{D}_x, Y_1, Y_2\}$. Then we will show that, with the only exception of a closed thin subset, on the level sets of I the restriction of $\{Y_1, Y_2\}$ form a solvable structure for the (restricted) one-dimensional distribution $\langle \tilde{D}_x \rangle$. Therefore the solutions of the given ODE which are in generic position can be described by the whole collection of generic solutions on each level set.

The level sets $\bar{\Sigma}_c := \{I = c : c \in \mathbb{R}\}$ are well defined on $M_1 = \mathcal{E} \backslash \Gamma$, with Γ the submanifold of $\mathcal{E}$ defined by condition $Ai(1,x)u - (u_1 + x)Ai(x) = 0$. We denote by Σ'_c the submanifolds of $\bar{\Sigma}_c$ defined by $Bi(x) + cAi(x) = 0$. On each $\Sigma_c = \bar{\Sigma}_c \backslash \Sigma'_c$ one can write u_1 in the following form

$$u_1 = \frac{Bi(1,x)u - Bi(x)x + cAi(1,x)u - cAi(x)x}{Bi(x) + cAi(x)}$$

and the restrictions $\tilde{D}_x^c = \tilde{D}_x|_{\Sigma_c}$ admit $\{Y_1^c = Y_1|_{\Sigma_c}, Y_2^c = Y_2|_{\Sigma_c}\}$ as a solvable structure defined on the whole Σ_c. Then, if on Σ_c we use the volume form $\bar{\Omega} = dx \wedge du \wedge dw$ and define

$$\bar{\Delta} = Y_1^c \lrcorner Y_2^c \lrcorner \tilde{D}_x^c \lrcorner \bar{\Omega}, \quad \bar{\omega}_1 = Y_2^c \lrcorner \tilde{D}_x^c \lrcorner \bar{\Omega} \quad \bar{\omega}_2 = Y_1^c \lrcorner \tilde{D}_x^c \lrcorner \bar{\Omega},$$

one gets that $\bar{\omega}_2 = dI_1$ with

$$I_1 = w + \ln|u| - \ln|Bi(x) + cAi(x)|.$$

Moreover, a further restriction to the level sets $\Sigma_{c,h_1} = \{I_1 = h_1 : h_1 \in \mathbb{R}\} \cap \Sigma_c$, entails that $\bar{\omega}_1|_{\Sigma_{c,h_1}}$ is the exterior derivative of

$$I_2 = \frac{ue^{-h_1}}{Bi(x) + cAi(x)} + e^{-h_1} \int \frac{x}{Bi(x) + cAi(x)} dx.$$

These computations give the generic solutions on each Σ_c in the implicit form $\{I_1 = h_1, I_2 = h_2\}$, with $h_1, h_2 \in \mathbb{R}$. Hence, by collecting all these solutions and writing them explicitly, one finally gets the following family of generic solutions

$$u = -(Bi(x) + CAi(x)) \int \frac{x}{Bi(x) + CAi(x)} dx,$$

with $C \in \mathbb{R}$.

Example 8. Consider the ODE $\mathcal{E}$ defined by

$$u_2 = -\frac{u_1}{x+u} + x(x+u) - \frac{2}{x+u}, \tag{4.11}$$

with $x + u \neq 0$. Equation (4.11) has no point symmetries but admits an exponential nonlocal symmetry in the λ-covering with $\lambda = 1/(x+u)$. In fact, by considering the covering system $\mathcal{E}'$ defined by (4.11) together with $w_1 = \lambda$, one gets a symmetry Y generated by $\varphi^1 = -(x+u)e^w$ and $\varphi^2 = e^w$.

As in the previous example, equations $L_{Y_3}(\mathcal{D}_2) \subset \mathcal{D}_2$ admit particular solutions in terms of Airy functions and we will use joint scalar invariants of $\{\tilde{D}_x, Y_1, Y_2\}$ to describe a family of generic solutions.

Here we use the following joint invariant

$$I = \frac{(-x-u)Bi(1,x) + (2+u_1)Bi(x)}{(x+u)Ai(1,x) - (2+u_1)Ai(x)}.$$

In this case, the level sets $\bar{\Sigma}_c := \{I = c : c \in \mathbb{R}\}$ are well defined on $M_1 = \mathcal{E} \backslash \Gamma$, with Γ the submanifold of $\mathcal{E}$ defined by condition $(x+u)Ai(1,x) - (2+u_1)Ai(x) = 0$. We denote by Σ'_c the submanifolds of $\bar{\Sigma}_c$ defined by $Bi(x) + cAi(x) = 0$. On each $\Sigma_c = \bar{\Sigma}_c \backslash \Sigma'_c$ one can write u_1 in the following form

$$u_1 = \frac{(x+u)(cAi(1,x) + Bi(1,x)) - 2(cAi(x) + Bi(x))}{Bi(x) + cAi(x)}$$

and the restrictions $\tilde{D}^c_x = \tilde{D}_x|_{\Sigma_c}$ admit $\{Y^c_1 = Y_1|_{\Sigma_c}, Y^c_2 = Y_2|_{\Sigma_c}\}$ as a solvable structure defined on the whole Σ_c. Then, if on Σ_c we use the volume form $\bar{\Omega} = dx \wedge du \wedge dw$ and define

$$\bar{\Delta} = Y^c_1 \lrcorner Y^c_2 \lrcorner \tilde{D}^c_x \lrcorner \bar{\Omega}, \quad \bar{\omega}_1 = Y^c_2 \lrcorner \tilde{D}^c_x \lrcorner \bar{\Omega} \quad \bar{\omega}_2 = Y^c_1 \lrcorner \tilde{D}^c_x \lrcorner \bar{\Omega},$$

one gets that $\bar{\omega}_2 = dI_1$ with

$$I_1 = w + \ln|x+u| - \ln|Bi(x) + cAi(x)|.$$

Moreover, a further restriction to the level sets $\Sigma_{c,h_1} = \{I_1 = h_1 : h_1 \in \mathbb{R}\} \cap \Sigma_c$, entails that $\bar{\omega}_1|_{\Sigma_{c,h_1}}$ is the exterior derivative of

$$I_2 = \frac{ue^{-h_1}}{Bi(x) + cAi(x)} + e^{-h_1} \int \frac{2(Bi(x) + cAi(x)) - x(Bi(1,x) + cAi(1,x))}{(Bi(x) + cAi(x))^2} dx.$$

These computations give the generic solutions on each Σ_c in the implicit form $\{I_1 = h_1, I_2 = h_2\}$, with $h_1, h_2 \in \mathbb{R}$. Hence, by collecting all these solutions and writing them explicitly, one finally gets the following family of generic solutions for

$$u = -(Bi(x) + CAi(x)) \int \frac{2(Bi(x) + CAi(x)) - x(Bi(1,x) + CAi(1,x))}{(Bi(x) + CAi(x))^2} dx$$

with $C \in \mathbb{R}$.

Acknowledgments

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